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Becoming a Mathematical PROBLEM SOLVER

Take a page from the humanities and have your students investigate mathematics in writing.

Nicole R. Rigelman

What *is* mathematics? What does it mean to *do* mathematics? The answers to these two questions critically influence the ways in which learners respond to a mathematical problem. On one hand, if they see questions as having only one correct answer and as needing to be solved in one particular way, they will be prone to treat mathematics as a rigid discipline. They will likely view it as a subject that one is either good at or not. On the other hand, if they see problems as opportunities to explore, create, and prove, they will experience the beauty and wonder of mathematics and realize that the unknown provides a chance for new learning.

Since the 1980s, numerous calls have been made for increased attention to problem solving in the curriculum (NCTM 1980, 1989, 2000; NRC 1989, 2001), yet problem solving continues to exist at the periphery of the school mathematics curriculum rather than being central to how mathematics learning is taking place (Hiebert et al. 1997; Hiebert et al. 2003). With the release of the Common Core State Standards for Mathematics (CCSSI 2010), we are reminded again that making sense of problems and persevering in solving them are important mathematical practices that we seek to develop in our students.

As a mathematics teacher, it is a struggle to help students see that problem solving is a process for learning mathematics and not an isolated task. I wanted my students to recognize that they were problem solving any time an immediate answer or solution path was not at hand. This goal led me to incorporate nonroutine problems regularly into the curriculum. One day, as I was distributing a problem, Chris asked, "Do we have to write this up so it can be scored?" At that moment, I realized that my goal was not achieved.

Both my students and I were focused on what I refer to now as problem *performing* instead of problem *solving,* the latter being the goal. Chris's question helped me realize that the focus of problem solving had been on scoring when the focus needed to be on the act of solving the problem and on learning from that process. My hope was that problem solving would allow students to discover mathematical relationships and pose new questions of their own. Something about

Resources for Worthwhile Mathematical Tasks

Driscoll, Mark. 1999. *Fostering Algebraic Thinking: A Guide for Teachers Grades 6–10.* Portsmouth, NH: Heinemann.

Driscoll, Mark, Rachel Wing DiMatteo, Johannah Nikula, and Michael Egan. 2007. *Fostering Geometric Thinking: A Guide for Teachers Grades 5–10.* Portsmouth, NH: Heinemann.

Small, Marian. 2009. *Good Questions: Great Ways to Differentiate Mathematics Instruction.* New York: Teachers College Press.

my practice needed to change. Problem performing needed to remain part of classroom practice but become less pronounced if students were to truly engage in—and not lose sight of—problem solving.

CONTEXT OF THE MATHEMATICS PROGRAM

Chris posed the question about "writing up the problem" in the spring of his seventh-grade year. The next year, I approached my instruction differently. This article is based on my experience teaching an eighth-grade heterogeneous mathematics class in a suburban middle school that used reform-oriented curriculum materials.

Even though these students regularly engaged in problem situations, these situations were not always problematic. At time, this occurred because the curriculum materials did too much to walk students through the process. At other times, students immediately recognized a problem as similar to others they had seen, so they could apply and refine experiences for this new problem. Although this situation is not bad, it does not allow students to engage in true problem solving. Basically, the students needed additional tasks beyond those in the curriculum materials if they were to engage in higher-level thinking and problem solving. I needed to provide tasks that would prompt students' own questions and inquisitiveness.

As various tasks were posed, students experienced increased discomfort because they did not know what to do or where to begin. Inspired by the writing process (i.e., prewrite, write, revise, edit, and publish) assigned by my humanities colleagues, a *Problem Solving Journal* became a vehicle for engaging students in the process of *doing* mathematics. Rather than placing the problem-solving focus on the end result, these journals were to engage students in a "prewrite" phase (Elbow 1998).

They were not to worry about taking every problem to the "publish" phase quite yet. I used this *Problem Solving Journal* to develop the habits of mind of a mathematician—using reasoning and sense making, modeling, looking for patterns, persevering, proving, and generalizing (Cuoco, Goldenberg, and Mark 1996). My inquiry into using the *Problem Solving Journal* to foster mathematical thinking and problem-solving habits of mind in my students resulted in redesigning the structures and processes I had used previously.

Before using the writing journal, my students were not engaging in a prewrite phase when confronted with a math problem. As they began work, they recorded what would be necessary in a final product; missteps or incorrect thinking were promptly erased. Elbow (1998), a writing professor, describes a parallel dilemma in writing and the need for free writing, a time to babble while disregarding grammar and spelling. At this early stage, students were free to explore a variety of possible solution paths and to write incorrect thinking or thoughts they might later abandon. They also needed to realize that not everything they recorded in their journal would necessarily be shared. They needed to know that tentative ideas should be explored.

They also needed to understand that it was acceptable to be stuck on a problem. As might occur with writing, the journal could be a place to record questions and frustrations as well as serve as a tool to help them get unstuck. Finally, their journals could be a place where they noted others' ideas that they wanted to remember, observed connections across problems, and described what-if questions. If they selected one of these problems to "publish," all elements would be available for students to draw on for their showcase problem.

CONFRONT EXISTING BELIEFS

To convince students that it was acceptable to work with a problem without necessarily finding a solution (Boaler 2002), I needed to better understand their beliefs about mathematics and what it means to do mathematics. Only then could these beliefs be changed.

When the school year began, I asked my students about math and what it means to *do* math. I then shared the diagram called A Mathematician's Work (see **fig. 1**). This diagram was inspired by the writing process described above and *Principles and Standards for School Mathematics* (NCTM 2000). A large copy of the diagram was posted on the wall so that students could reference various components throughout the year. It was my hope that as these behaviors were named and clearly valued, students would grow to see themselves as mathematicians.

EXPLORE WORTHWHILE PROBLEMS

The next step was selecting math tasks designed to encourage exploration and prompt the development of models,

conjectures, and generalizations. The problems needed to be interesting to students and worthy of their time. Sometimes these problems came directly from the curriculum materials; other problems were adapted from these materials or from other resources (a list can be found in **the sidebar** on p. 418). The tasks typically connected to the content of the current unit or the unit we were finishing.

The problems were photocopied and prepared so students could glue them directly into their journals. On alternating Mondays, I gave students eight to twelve minutes to freely write on a new math problem. They made observations and sketches, asked what-if questions, and listed "I wonder" statements and/or connections they saw. This time was intentionally short so that students needed to continue working on the problem outside of class. During the two weeks, if students ever finished class work early, I encouraged them to discuss with other early finishers their thoughts and ideas and any relationships they discovered in their investigation of the problem. My colleagues often told me that similar conversations took place during free time in other classes. These journals encouraged students to jot down ideas along the way, free from concerns about a final draft.

On the second Monday, students had about fifteen minutes to share their draft solution with a peer or small group. While students shared the draft, I listened, identified the strategies (make a systematic list, draw a diagram, use guess and check, and so on) that students used when appropriate, and recorded the strategy on a sentence strip if it was a new strategy. Over time, the sentence strips formed a bulletin board of problem-solving strategies and served as a reminder for students of possible strategies they could try if they were stuck.

Fig. 1 This cycle of mathematics problem solving, posted on the wall for reference, was meant to help students internalize what mathematicians do.

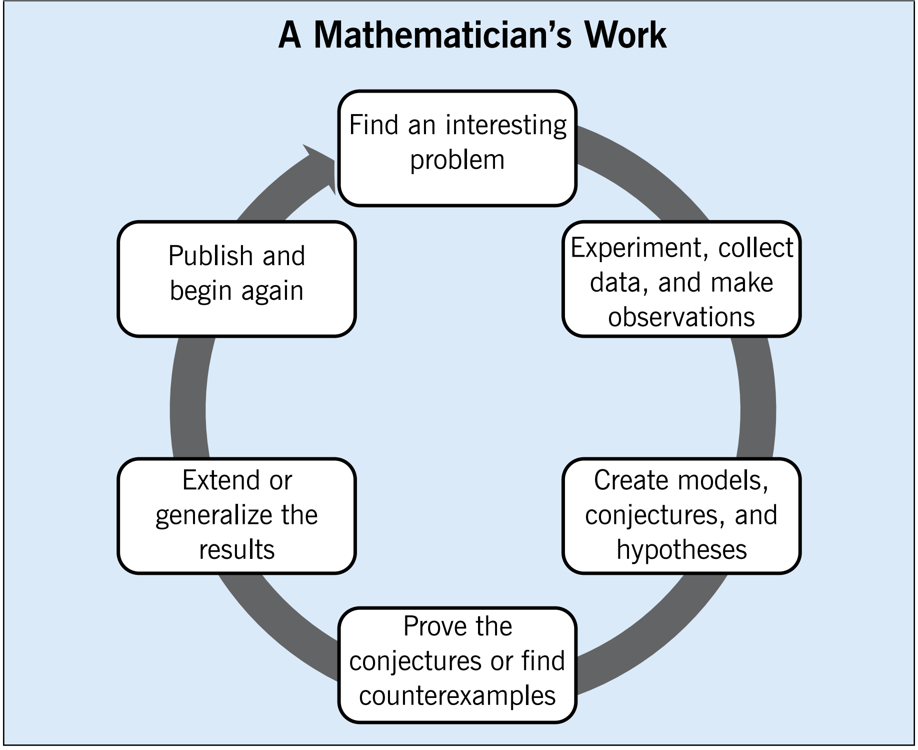


Fig. 2 The Baby Limitation problem provided a context for students to explore different problem-solving strategies.

In one society, the government makes the following decree concerning family size:

a. Each family shall continue to have children until a girl is born.
b. As soon as a girl is born, the family shall have no additional children.

Assuming this policy has been in effect for a long time, find the average number of children per family. Also, find the ratio of females to males in this society.

Source: Adapted from Konold (1994)

This move also supported the development of a common language for talking about strategies. Finally, students reflected on their small-group discussion, made note of new strategies they wanted to remember, analyzed the effectiveness of various strategies, and determined what they believed would be a quality response to the problem.

The Baby Limitation problem (see **fig. 2**) provided a context for this discussion and a view into students' potential responses. Konold suggests that because the results of this problem are counterintuitive, it increases students' interest and motivation (Konold 1994). Typically, students begin with the assumption that there is a 50-50 chance of a couple having a boy or girl. They generate data, often by flipping a coin, and track family size. Variation occurs with the number of families that students decide they need to simulate. Their sample sizes vary anywhere from 20 to 100. Finally, students draw

Fig. 3 This scoring guide, given to students, was used to evaluate their thinking.

Review the criteria below that are relative to your work on the current set of journal problems. Place a check mark in front of each item that is reflected in your work on these problems.

EXCEPTIONAL

Work reflects all the criteria of quality work. Each problem goes above and beyond in two or more of the following ways.

_____ Provides multiple approaches for solving the problems.
_____ Describes conjectures and/or makes generalizations.
_____ Offers observations about the mathematical relationships.
_____ Poses and investigates "what if" questions and "I wonder" statements.
_____ Gives examples and/or counterexamples to illustrate ideas.
_____ Justifies why concepts and strategies are effective and reasonable.

QUALITY

Work shows solid understanding of the problems' mathematical ideas.

_____ Uses an appropriate and systematic strategy for solving the problem.
_____ Represents solution process with diagrams/sketches, words, and/or numbers and math symbols.
_____ Communicates mathematical reasoning coherently and precisely.

IN DEVELOPMENT

Work shows nearly complete understanding of the problems' mathematical ideas.

_____ Uses a strategy that is only partially useful or partially complete.
_____ Gives clear evidence of the solution process, and the process is complete or nearly complete and systematic.
_____ Provides reasonably clear explanations and/or descriptions.

RESTART/RETHINK

Work shows misunderstanding or no understanding of the problems.

_____ Uses inappropriate or incomplete strategies.
_____ Provides sketchy, ineffective, or no explanation.

Sources: Adapted from *Follow-Up Assessment Form* (Foreman 1996) and *Mathematics Problem Solving Scoring Guide* (Oregon Department of Education 2011)

conclusions from their data (e.g., determine the average number of children in a family and the ratio of boys to girls in this society), and some will summarize the information they gather using such representations as frequency charts, bar graphs, and circle graphs.

One student noticed that "one child families make up the largest population" and "since I did 50 trials to get a girl, there are 50 girls." After hearing statements like these, I posed questions to find out what she knew and understood about the relationships she had noticed (e.g., "Can you make a generalization?") and prompted her to ask her own what-if questions.

DEVELOP CRITERIA FOR QUALITY WORK

When students have a strong sense about what counts as high-quality work, they will be better able to produce it. After students used this problem-solving format for three rounds, we collaboratively generated a scoring guide that they could use while solving a problem. When asked what a quality response looks like, students brainstormed the typical attributes: clear communication; correct strategies and solutions; and multiple representations such as diagrams, models, tables, graphs, or formulas.

If students did not include important criteria, such as "justify *why* concepts and strategies are effective and reasonable," I suggested the idea or shared a student work sample that illustrated these attributes. Involving students in this process helped them internalize the criteria for high-quality work. They then used this list to reflect on the quality of their own work and any revisions or additions they might want to make. After asking students to define general features of quality work, I followed up by asking what more would be needed to create *exceptional* work. The criteria were very close to those shown in **figure 3**, which represents a compilation of ideas from students as well as our state assessment scoring guide. Students used these criteria to assess the work in their *Problem Solving Journals*.

ENCOURAGE UNCERTAINTY AND WONDER

The selection of tasks was a key component in this process. Students needed to engage in rich problems that had multiple entry points and that connected to the mathematical trajectory of the curriculum. Problems that students solved that were loosely related to problems already explored in class presented opportunities for them to make connections between strategies and or representations, even when there was no clear pathway to a solution. This scenario allowed them to dive in to the task with more confidence than might otherwise occur. For example, a number of students made a connection between the Baby Limitation problem and a binomial experiment that we

had explored (Checker-A game and Checker-B game from Shaughnessy and Arcidiacono 1993).

After seeing this connection, one student did not seem overly challenged by the ideas in this problem. Her work initially included a tree diagram, with a short answer for the average number of children per family and the boy-to-girl ratio (see **fig. 4**). After discussing the task with a partner, she became much more engaged. Because her partner needed help to make sense of her tree diagram, she then created the data table to summarize what happened as the size of the family increased, commenting on the likelihood of the family makeup (number of boys and number of girls) per 100 families (see **fig. 5**). She also examined what would happen if she simulated this situation through coin tosses (see the data in **fig. 6**) and then compared these results with the theoretical results. Discussions with a partner, in addition to the expectation that students select a problem to "publish," encouraged this student to investigate what-if questions that pressed her to make conjectures, analyze patterns, and offer generalizations.

This student's actions revealed potential limits of her understanding about the relationship between experimental and theoretical probability. Her initial work was brief yet solid. She delved deeper into what the data revealed about larger families (see **fig. 5**) and examined her data from the simulation (see **fig. 6**, comments that 9 or more children "won't happen with only 100 families"). Then she was ready to push aside data from families with more than 8 children. She explained that the larger families were not coming up in her simulation and that 8 or more children in a family represented less than 1 out of 100 families. Without this further development, I would not have known about this student's understanding, and her limited conceptions would not have been evident.

Fig. 4 One student constructed this tree diagram as a solution to the Baby Limitation problem. Her partner then asked her to explain her work in another way.

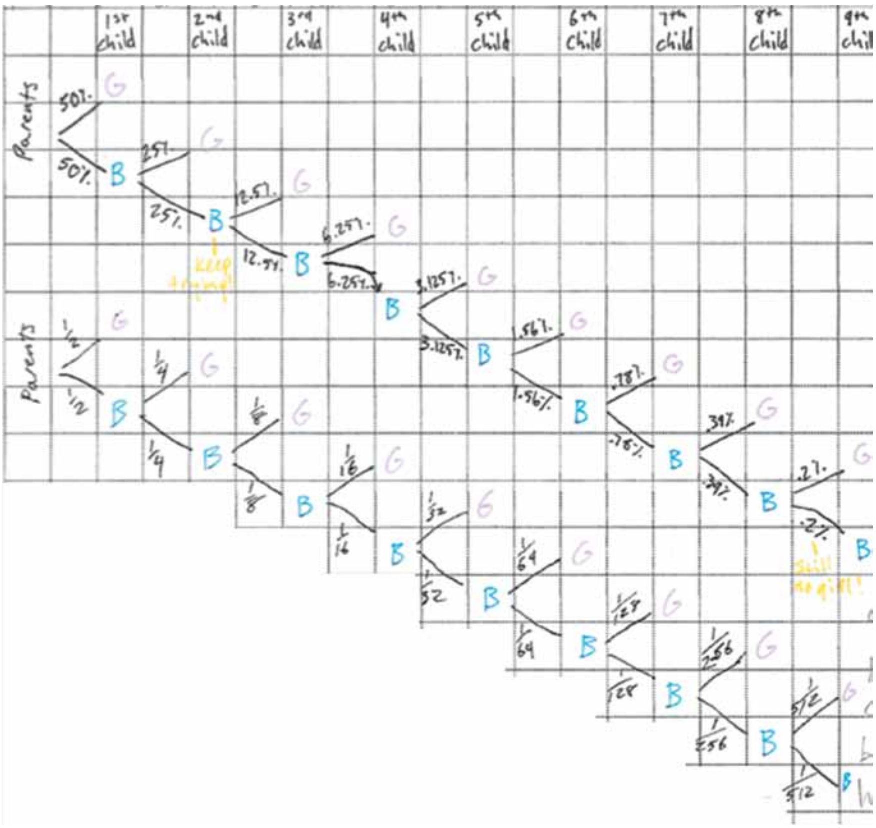


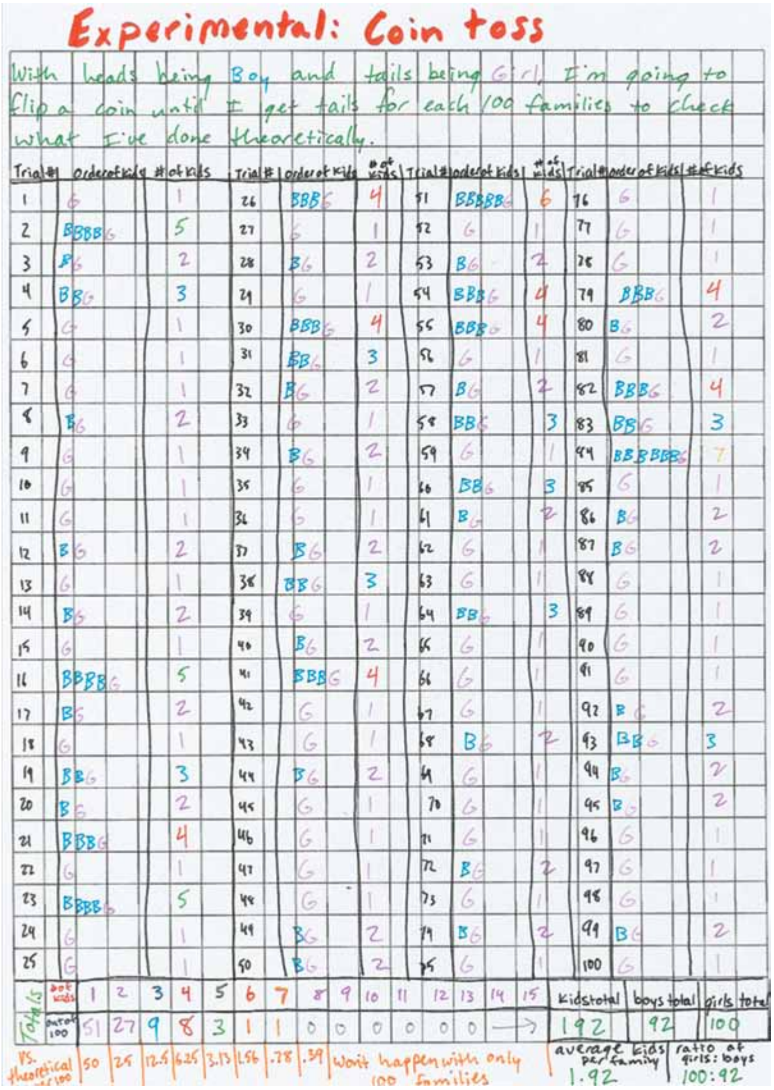
Fig. 5 This theoretical data table, produced after the tree diagram, was the student's attempt to explain her solution to her partner.

Order of kids	# of kids	# of boys	# of girls	% of families	# of kids per 100 families	# of boys per 100 families	# of girls per 100 families
G	1	0	1	50%	50	0	50
BG	2	1	1	25%	50	25	25
BBG	3	2	1	12.5%	37.5	25	12.5
BBBG	4	3	1	6.25%	25	18.75	6.25
BBBBG	5	4	1	3.125%	15.625	12.5	3.125
BBBBBG	6	5	1	1.563%	9.375	7.813	1.563
BBBBBBG	7	6	1	.7813%	5.469	4.688	.7813
BBBBBBBG	8	7	1	.391%	3.125	2.734	.391
BBBBBBBBG	9	8	1	.195%	1.758	1.56	.195
BBBBBBBBBG	10	9	1	.098%	.977	.879	.098
BBBBBBBBBBG	11	10	1	.049%	.537	.488	.049
BBBBBBBBBBBG	12	11	1	.024%	.293	.269	.024
BBBBBBBBBBBBG	13	12	1	.012%	.159	.146	.012
BBBBBBBBBBBBBG	14	13	1	.006%	.085	.079	.006

total % 99.8053%
children per avg. family 1.9785

total % 99.9993%
# of children per avg. family 1.99903

Fig. 6 The experimental data from simulating the Baby Limitation problem helped solidify understanding and revealed some limits on that understanding.


Experimental: Coin toss

With heads being Boy and tails being Girl I'm going to flip a coin until I get tails for each 100 families to check what I've done theoretically.

Trial#	order of kids	# of kids	Trial#	order of kids	# of kids	Trial#	order of kids	# of kids	Trial#	order of kids	# of kids
1	G	1	26	BBBG	4	51	BBBBBG	6	76	G	1
2	BBBBG	5	27	G	1	52	G	1	77	G	1
3	BG	2	28	BG	2	53	BG	2	78	G	1
4	BBG	3	29	G	1	54	BBBG	4	79	BBBG	4
5	G	1	30	BBBG	4	55	BBBG	4	80	BG	2
6	G	1	31	BBG	3	56	G	1	81	G	1
7	G	1	32	BG	2	57	BG	2	82	BBBG	4
8	BG	2	33	G	1	58	BBG	3	83	BBG	3
9	G	1	34	BG	2	59	G	1	84	BBBBBBG	7
10	G	1	35	G	1	60	BBG	3	85	G	1
11	G	1	36	G	1	61	BG	2	86	BG	2
12	BG	2	37	BG	2	62	G	1	87	BG	2
13	G	1	38	BBG	3	63	G	1	88	G	1
14	BG	2	39	G	1	64	BBG	3	89	G	1
15	G	1	40	BG	2	65	G	1	90	G	1
16	BBBBG	5	41	BBBG	4	66	G	1	91	G	1
17	BG	2	42	G	1	67	G	1	92	BG	2
18	G	1	43	G	1	68	BG	2	93	BBG	3
19	BBG	3	44	BG	2	69	G	1	94	BG	2
20	BG	2	45	G	1	70	G	1	95	BG	2
21	BBBG	4	46	G	1	71	G	1	96	G	1
22	G	1	47	G	1	72	BG	2	97	G	1
23	BBBBG	5	48	G	1	73	G	1	98	G	1
24	G	1	49	BG	2	74	BG	2	99	BG	2
25	G	1	50	BG	2	75	G	1	100	G	1

Totals	# of kids	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Kids total	boys total	girls total
	out of 100	51	27	9	8	3	1	1	0	0	0	0	0	0	→		192	92	100
	vs. theoretical out of 100	50	25	12.5	6.25	3.13	1.56	.78	.39	Won't happen with only 100 families							average kids per family 1.92	ratio of girls : boys 100:92	

PUBLISH AND BEGIN AGAIN

As mentioned previously, students received a new problem every two weeks throughout the year. At midterm and at the end of the term, as with the writing process, students reviewed their *Problem Solving Journals* to select a problem that they would like to revisit for publication and to self-assess using the Mathematics Problem Solving Scoring Guide (Oregon Department of Education 2011). This guide gave students a score for each dimension instead of one overall score as with the previous guide (see **fig. 3**). When they self-assessed, they recorded both a score and the reasoning behind it, citing specific evidence.

The excerpts in **figures 4–6** provide evidence of the final product one student created and then self-assessed. Students were asked to justify that these were the accurate scores. Students also switched papers with a partner who assessed his or her work, recording both a score and the reasoning behind it, and then the pair met to discuss both items. The pair came to an agreement about the scores, and recorded this information on the work. After this feedback, students were allowed to revise their work. Finally, students submitted their polished work for evaluation and inclusion in their portfolio, then shared with the parents during student-led conferences.

THE POWER OF THE JOURNALS

As I used this *Problem Solving Journal* and the associated processes and structures, I discovered that over time, most students' beliefs about mathematics and what it means to *do* mathematics began to shift. The journal gave students a place in which they felt free to explore. It helped them see connections across problems and document what they were learning from others. Because their work on the problems began as a draft, they could add notes about others' thinking when ideas were prompted from their conversations with others. As noted in the example, sometimes they responded by determining that they needed to clarify their work for others.

The structure of publishing their work later motivated students to take the feedback they had gathered from others to create their own best work. Over time, the *what if* questions and *I wonder* thoughts that students pursued became more widespread, rather than isolated to a few. Students developed a view of themselves as problem solvers who needed to make sense of the mathematics so that they could work toward true understanding.

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